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A test of the method of Fink & Soh for following vortex-sheet motion

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Fink & Soh (1978) reported a technique to calculate numerically the motion of vortex sheets. They claim it gives reliable results. This paper re-examines the error in the calculation of the velocity of mesh points representing the sheet and shows that the test case used by Fink & Soh is not an adequate one. Instead the roll-up of a vortex sheet shed by a ring wing is studied. The results obtained proved unreliable. (Although several possibilities are discussed) the reason for the breakdown in results remains unknown.

1. Introduction

The roll-up of a vortex sheet in an ideal fluid is a problem that has received considerable attention over the years (see a recent review by Saffman & Baker 1979). A particular frustration has been the difficulty in obtaining reliable numerical results. With the advent of high-speed computers there have been several attempts to study vortex sheet roll-up using a discretization first introduced by Rosenhead (1931). He replaced the vortex sheet by a finite number of point vortices and considered their subsequent motion as marking out the vortex sheet. When the number of point vortices was substantially increased, an unsatisfactory feature of the results consistently emerged. The motion of the point vortices becomes chaotic in the region of the vortex sheet roll-up.

Different modifications have been incorporated to regularize the solution. Kuwahara & Takami (1973) and Chorin & Bernard (1973) used discrete vortices to calculate the vortex sheet velocity, but there is no adequate accounting of the errors introduced by these modifications. Moore (1974) on the other hand recognized the inadequacy of representing the innermost terms of the spiral with a finite number of point vortices and instead used a central point vortex which grew by amalgamation. His results showed better agreement with the similarity solution found by Kaden (1931). However, he observed an instability developing on the outermost turn of the spiral and it is uncertain whether its origin is numerical or physical.

Fink & Soh (1978) are the first authors to present error estimates for the point vortex approximation to a vortex sheet. In particular, they show that, unless the vortices are evenly spaced in arclength, the error in calculating the velocity at one of the vortices is $O(\ln h_1/h_2)$, where h_1 and h_2 are the adjacent spacings with its neighbours. This suggests that, as the vortex points move and lose their uniform spacing, the error in calculating their velocities will grow and eventually destroy the description of the

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vortex sheet. Fink & Soh (1978) therefore propose a method whereby the vortices are continually redistributed to ensure uniform spacing. They applied their method to the roll-up of a vortex sheet shed by an elliptically loaded wing, but included a single-vortex representation for the inner region of the spiral. They claim their results demonstrate the reliability of their technique.

In support of their claim they have made estimates of the error in calculating the sheet velocity. However, their estimates are misleading and one purpose of this paper is to provide more careful error estimates. If the sheet forms an open curve, for example the sheet rolling up behind an elliptical loaded wing, the error becomes large in the central region of the spiral even if infinitely many evenly spaced points could be chosen to represent the sheet. It is difficult to assess the errors if a finite number of points are chosen to represent the sheet together with a single-vortex approximation for the inner region of the spiral. This case, therefore, does not provide a straightforward test of their method.

On the other hand, there is no such difficulty for the vortex sheet forming a closed curve, for example the vortex sheet shed by a ring wing, and this case is studied as a definite test of the method. The method fails to produce reliable results; the sheet crosses itself. Although several possibilities are discussed, no definitive reason is found for the breakdown. Fundamental questions remain about the nature of vortex sheets and whether numerical methods can adequately treat their motion.

2. Error analysis for velocity calculation

For an ideal fluid containing a vortex sheet, the complex-conjugate velocity field, $\bar{q} = u - iv$, can be expressed as an integral of the vortex-sheet strength $\gamma(s)$ along the sheet, $z(s) = x(s) + iy(s)$, which is parametrized by its arclength s,

$$\bar{q} = \frac{1}{2\pi i}\int \frac{\gamma(s)\,ds}{z - z(s)} + \bar{Q}, \tag{2.1}$$

where Q is an irrotational velocity field that takes into account the presence of boundaries and any other external flow. It is assumed that the velocity field Q is known or that it may be calculated accurately without difficulty, and so may be ignored for our purposes. The velocity of the sheet (at $z(s_0)$ say) is given by the principal-value integral

$$q(s_0) = \frac{1}{2\pi i}\,\mathcal{P}\!\!\int \frac{\gamma(s)\,ds}{z(s_0) - z(s)}. \tag{2.2}$$

First consider the sheet as a closed curve. The integrand in (2.2) is now periodic and the integral may be written as

$$\bar{q}(s_0) = \frac{1}{2\pi i}\,\mathcal{P}\!\!\int_{s_0-L}^{s_0+L} \frac{\gamma(s)}{z(s_0) - z(s)}\,ds, \tag{2.3}$$

where $2L$ is the period. Subtracting from (2.3) the integral

$$-\frac{i}{2\pi i}\,\mathcal{P}\!\!\int_{s_0-L}^{s_0+L} \frac{\gamma(s_0)\,ds}{z'(s_0)\,(s - s_0)} = 0, \tag{2.4}$$

a new integrand, $f(s)$, is obtained, where

$$2\pi i f(s) = \frac{\gamma(s)}{z(s_0)-z(s)} + \frac{\gamma(s_0)}{z'(s_0)\,(s-s_0)}. \tag{2.5}$$

(The prime refers to differentiation with respect to s.)

This integrand has a removable singularity at s_0 and no other singularities provided the sheet is smooth, i.e. it has a continuous tangent, and γ has no singularities. In fact, we assume that $f(s) \in c^{2m+1}[-L, L]$ and we introduce a mesh $s = \{s_0 + nh\}$, $-N \leqslant n \leqslant N$, $Nh = L$.

The Euler–McLaurin summation formula yields the following expression:

$$\int_{s_0-L}^{s_0+L} f(s)\,ds = \frac{h}{2}\{f(s_0-L)+f(s_0+L)\} + h\sum_{n=N-1}^{N+1} f(s_0+nh)$$
$$+ \sum_{p=0}^{m} c_p h^{2p+2}\{f^{(2p+1)}(s_0+L) - f^{(2p+1)}(s_0-L) + c_m h^{2m+2} f^{(2m+1)}(\xi), \tag{2.6}$$

where c_p are constants independent of h and $f(s)$. The bracketed superscript refers to the order of differentiation with respect to s and $\xi \in [s_0 - L, s_0 + L]$. The first two terms on the right-hand side are the trapezoidal approximation to the integral. Since the first term of $f(s)$ is periodic and $1/(s-s_0)$ is an odd function about s_0, the trapezoidal approximation becomes

$$q(s_0) = \frac{h}{2\pi i}\sum_{\substack{n=-N \\ n\neq 0}}^{N-1} \frac{\gamma(s_0+nh)}{z(s_0)-z(s_0+nh)} + hf(s_0), \tag{2.7}$$

where

$$f(s_0) = \frac{1}{2\pi i z'(s_0)}\left\{-\gamma'(s_0) + \frac{\gamma(s_0)\,z''(s_0)}{2z'(s_0)}\right\}. \tag{2.8}$$

This expression may be simplified by using the identities $z'\bar{z}' = 1$ and $\bar{z}''z' + z''\bar{z}' = 0$. Thus

$$f(s_0) = -\frac{1}{2\pi i}\{\gamma'(s_0)\,\bar{z}'(s_0) + \tfrac{1}{2}\gamma(s_0)\,\bar{z}''(s_0)\}. \tag{2.9}$$

The source of errors in the approximation, equations (2.7) and (2.9), is twofold; there is an error in approximating $f(s_0)$ by finite differences and there are additional terms from (2.6), namely

$$\sum_{p=0}^{m} c_p h^{2p+2}\{f^{(2p+1)}(s_0+L) - f^{(2p+1)}(s_0-L)\} + c_m h^{2m+2} f^{(2m+1)}(\xi). \tag{2.10}$$

The periodicity of the first term in $f(s)$ and the evenness of the odd derivatives of $1/(s-s_0)$ about s_0 ensure there is no contribution to the sum in (2.10). The dominant contribution to the error will be due to the finite-difference approximation of (2.9). If standard central differences are used, for example

$$z'(s_0) = \{z(s_0+h) - z(s_0-h)\}/2h, \tag{2.11}$$

$$z''(s_0) = \{z(s_0+h) - 2z(s_0) + z(s_0-h)\}/h^2, \tag{2.12}$$

and a similar differencing for $\gamma'(s_0)$, the error in using (2.7) and (2.9) will be $O(h^3)$.

Fink & Soh (1978) use equal spacing in chordlength instead of arclength, which introduces a small error, but a far more serious error is their neglect of the curvature of the sheet in calculating the principal value of the integral. They have missed the second term in (2.9) and so the error in their calculations is $O(h)$ and not $O(h^3)$ as they claim.

We turn now to the more complicated case where the vortex sheet is an open curve. If the open curve has an end-point (at $z(0)$ say), the velocity is finite there only if $\gamma(0) = 0$.

In fact when $s_0 = 0$ the integrand in (2.2) is finite everywhere and the trapezoidal approximation may be used with an error $O(h^2)$. However, for points near the end-point, the error is $O(h)$. This is most easily seen by considering the error at $s_0 = h$ (i.e. the point adjacent to the end-point). For convenience, consider the sheet as having finite length L, for, if not, we run into the difficulty of representing the sheet by a finite number of points. As before, our approximation to $\bar{q}(h)$ will be

$$\frac{h}{2\pi i}\sum_{n=2}^{N-1}\frac{\gamma(nh)}{z(h)-z(nh)}-\frac{h}{2\pi i}\{\gamma'(h)\bar{z}'(h)+\tfrac{1}{2}\gamma(h)\bar{z}''(h)\},$$

where $Nh = L$. We obtain the asymptotic behaviour of the error by applying the Euler–McLaurin formula (see equation (2.6)) on $[0, L]$, where $f(s)$ is given by (2.5). Since

$$\int_0^L f(s)\,ds = \bar{q}(h)+\frac{\gamma(h)}{2\pi i z'(h)}\log\left|\frac{L-h}{h}\right|,$$

$$\frac{h}{2}\{f(0)+f(L)\}+h\sum_{n=1}^{N+1}f(nh) = \frac{h}{2\pi i}\sum_{n=2}^{N-1}\frac{\gamma(nh)}{z(h)-z(nh)}-\frac{h}{2\pi i}\{\gamma'(h)\bar{z}'(h)+\tfrac{1}{2}\gamma(h)\bar{z}''(h)\}$$
$$+\frac{\gamma(h)}{2\pi i\, z'(h)}\left\{-\tfrac{1}{2}+\sum_{n=2}^{N-1}\frac{1}{n-1}+\frac{1}{2(N-1)}\right\}$$

and $f^{(2p+1)}(0)$ and $f^{(2p+1)}(L)$ are $O(1)$, the error is

$$\frac{h\gamma'(0)}{2\pi i\, z'(0)}\left\{-\tfrac{1}{2}+\sum_{n=1}^{N-2}\frac{1}{n}-\log|N-1|\right\}+O(h^2) = \frac{h\gamma'(0)}{2\pi i\, z'(0)}\{\bar{\gamma}-\tfrac{1}{2}\}+O(h^2),$$

where $\bar{\gamma}$ is the Euler–Mascheroni constant. Thus the error is $O(h)$. If we consider points further away from the end-point, the $O(h)$ contribution to the error decreases and the error behaves more like $O(h^2)$.

If the open curve has no end-points (and so is infinitely long), we employ the Euler–McLaurin sum formula on an interval $[s_0 - L, s_0 + L]$, where L may be chosen arbitrarily. The sum in (2.10) yields a contribution to the error which, in the limit as $L \to \infty$, is $O(h^2/|z(s_0)-z(\pm\infty)|^2)$.

These results indicate the difficulty in approximating an infinite vortex sheet which is rolled up into a spiral. First, a finite number of mesh points must be used and the error in the velocity at the terminal mesh point is difficult to assess. Moreover, as more points are added to the central region of the spiral, the error in their velocities becomes larger.

Because of the complexities associated with a sheet of infinite length, this case does not provide an adequate test of the method proposed by Fink & Soh (1978). Instead

N	Calculated u	Error in u	Calculated v	Error in v
	(*a*) Circular sheet: exact velocity, $q = 0{\cdot}2165063 - 0{\cdot}125i$			
5	0·2129180	−0·003588	−0·1238720	0·001128
10	0·2160345	−0·000460	−0·1248536	0·000146
20	0·2164471	−0·000059	−0·1249822	0·000018
40	0·2164967	−0·000010	−0·1249989	0·000001
80	0·2165025	−0·000004	−0·1250024	−0·000002
	(*b*) Elliptical sheet: exact velocity, $q = 0{\cdot}6477635 - 0{\cdot}5300051i$			
10	−0·6465092	0·001254	−0·5326822	−0·002677
20	−0·6476290	0·000134	−0·5303204	−0·000315
40	−0·6477448	0·000019	−0·5300345	−0·000029

TABLE 1. Calculated velocity and errors for varying numbers of intervals.

a vortex sheet forming a closed curve is studied where the errors in calculating the velocity are known precisely.

Before giving the details of this study, the error analysis is checked for two cases where the velocities and hence vortex-sheet strength are both known instantaneously. At this stage the motion of the sheet is unimportant; the check is to see how accurately the velocity is calculated by a mesh evenly distributed in arclength. The central-difference approximations, (2.11) and (2.12), are used in conjugation with (2.9) and (2.7) as the test method.

A combination of velocity potentials inside and outside a unit circle is chosen to give a circular vortex sheet of strength $\gamma(s) = \cos(s)$, where $s = \theta$. The results of calculating the velocity at $s = -\frac{1}{6}\pi$ are presented in table 1, part (*a*), for varying numbers of intervals, N. Each time N is doubled, the error decreases by a factor 8, confirming the $O(h^3)$ error estimate. For $N \geqslant 40$, the relative error is 10^{-4} and is the limit obtainable using single precision on an IBM/370/158. The other case is an elliptical sheet, $\frac{1}{2}x^2 + y^2 = 1$, enclosing a stagnant core. The velocity potential for a rotating flow outside the ellipse can be calculated using elliptical co-ordinates. A particular solution was chosen corresponding to a vortex sheet of strength

$$\gamma = -2/\{[(x-1)^2+y^2]^{\frac{1}{4}}\,[(x+1)^2+y^2]^{\frac{1}{4}}\}.$$

Table 1, part (*b*), shows typical results. Again good agreement with the error analysis is found.

3. Numerical technique and results

The details of the method used to study the motion of a vortex sheet forming a closed curve differ in slight but important ways from the general method adopted by Fink & Soh (1978). In particular the accuracy is improved by keeping the points evenly spaced in arclength and by calculating the velocity of the sheet with an error $O(h^3)$ as described in the previous section. The method falls into three phases; there is the time advancement of the mesh points to ensure even spacing along the curve and finally the calculation of the change in the vortex-sheet strength.

First we describe the technique to redistribute the mesh points. Suppose that there are $N+1$ mesh points, $\{z_i(t)\}$, representing the vortex sheet at a given time t with $z_{N+1}(t) = z_1(t)$ since the sheet is a closed curve. The chordlength between mesh points is calculated and the total chordlength $\{\lambda_i\}$ measured from $z_1(t)$ defines a parametrization for the sheet:

$$\lambda_i = \sum_{j=1}^{i-1} |z_{j+1}(t) - z_j(t)|, \quad \lambda_1 = 0. \tag{3.1}$$

Thus $\{z_1(t)\}$ is known as a discrete function of $\{\lambda_i\}$. An approximate continuous function is obtained by using spline interpolation. The derivatives $\{dz_i/d\lambda\}$ are determined as the derivatives of the spline interpolation function at equal intervals in λ, $\{\lambda_{ei}\}$, and the arclength $\{s_i\}$ is calculated from

$$s = \int_0^\lambda \left|\frac{dz}{d\lambda}\right| d\lambda \tag{3.2}$$

using Simpson's rule. It is preferable to evaluate $\{dz_i/d\lambda\}$ at least at twice as many places as there are mesh points $\{z_i(t)\}$ to determine the arclength $\{s_i\}$ at a sufficient number of points. The arclength $\{s_i\}$ is now known as a function of $\{\lambda_{ei}\}$. This relationship is inverted and a spline interpolation used to determine $\{\lambda_{si}\}$ corresponding to even spacing in arclength, $\{iS_{N+1}/N\}$. Finally, evenly spaced points on the sheet $\{z_1^*(t)\}$ are obtained by interpolating $\{z_i(t)\}$, $\{\lambda_i\}$ and $\{\lambda_{si}\}$. Similarly any other variables, e.g. circulation, velocity, known at either $\{z_1(t)\}$ or $\{z_1^*(t)\}$ may be interpolated using $\{\lambda_i\}$ and $\{\lambda_{si}\}$. The accuracy in this procedure is limited by the choice of the spline interpolation. Since the curve is closed, a spline function which is periodic may be used with an error of $O(h^4)$. The derivative of the spline function gives an $O(h^3)$ approximation to the derivatives $\{dz_i/d\lambda\}$ and so the redistribution process is expected to have an error of $O(h^3)$.

Modified Euler integration forms the basis by which the mesh is advanced throughout a timestep Δt. The mesh points $\{z_i(t)\}$ are considered Lagrangian particles whose velocities are given by $q(z(s,t))$ from (2.2). Starting with $\{z_i\}$ evenly spaced along the vortex sheet at time t, a first approximation to the new position of the vortex sheet is

$$z_{1i} = z_i(t) + q(z_i(t))\,\Delta t, \tag{3.3}$$

where $q(z_i(t))$ is approximated by (2.7) and (2.9). An improved estimate is

$$z_{2i} = z_i(t) + \tfrac{1}{2}\{q(z_i(t)) + q(z_{1i})\}\,\Delta t. \tag{3.4}$$

To maintain $O(h^3)$ accuracy, the velocities $q(z_{1i})$ must be calculated as follows. A redistributed mesh $\{z_{ti}\}$ based on the approximation $\{z_{1i}\}$ is introduced, the velocities $q(z_{ti})$ at $\{z_{ti}\}$ are calculated from (2.7) and (2.9) and finally $q(z_{1i})$ is obtained by interpolating $q(z_{ti})$, $\{z_{ti}\}$ and $\{z_{1i}\}$.

To complete the update in time the mesh $\{z_{2i}\}$ must be redistributed to give $\{z_i(t+\Delta t)\}$. If only a simple Euler integration is performed, e.g. (3.3), $\{z_{1i}\}$ must be redistributed to give $\{z_i(t+\Delta t)\}$. After several time-steps have been computed, the error from simple Euler integration is $O(\Delta t)$, whereas for modified Euler integration it is $O(\Delta t^2)$.

In order to calculate the velocity, $q(z_i(t))$, it is necessary to know $\gamma(z_i(t))$ at the evenly spaced points and these values may be determined as follows. Since $\gamma = d\Gamma/dS$,

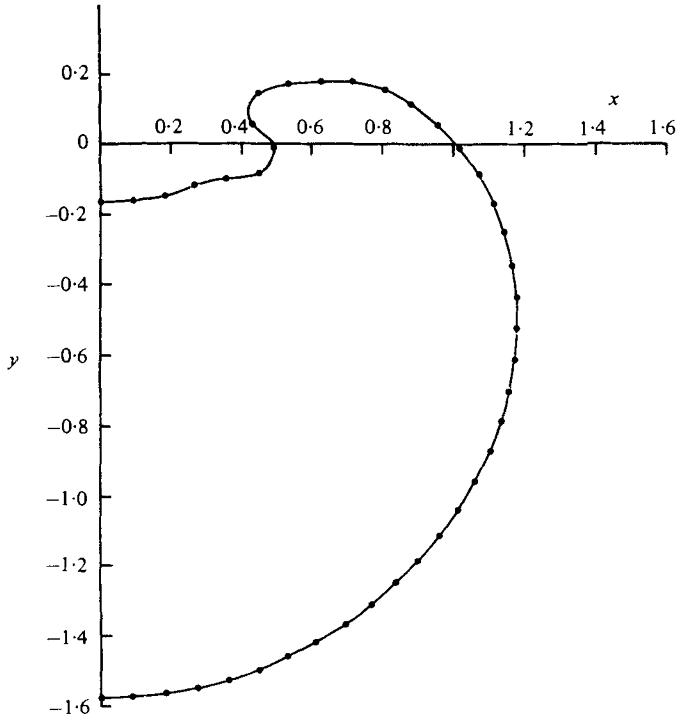

FIGURE 1. The vortex sheet behind a ring wing at $t = 1{\cdot}55$ with 41 mesh points along the sheet.

where $\Gamma(s)$ is the total circulation measured along the vortex sheet from some reference point, z_1 say, a central-difference formula may be used,

$$\gamma(z_i(t)) = (\Gamma(s_{i+\frac{1}{2}}) - \Gamma(s_{i-\frac{1}{2}}))/h, \tag{3.5}$$

where $\Gamma(s_{i\pm\frac{1}{2}})$ are the values of the circulation at the points located midway between the evenly spaced mesh points $\{z_i\}$. If $\Gamma(s)$ is known at $\{z_i\}$, $\Gamma(s_{i\pm\frac{1}{2}})$ is easily obtained by interpolation. This is analogous to the procedure Fink & Soh (1978) used. However, if spline interpolation is used during the interpolation of the circulation to be described below, spline derivatives may be used in place of (3.5), which improves the order of accuracy.

In both cases, $\Gamma(z_i(t))$ must be known. Thus the procedure in determining the change in $\gamma(z_i(t))$ depends on the ability to update $\Gamma(z_1(t))$. Since the mesh points are advanced in time as Lagrangian particles, $\Gamma(z_{2i}) = \Gamma(z_i(t))$ (or $\Gamma(z_{1i}) = \Gamma(z_i(t))$ if simple Euler integration is used or when the first step of the modified Euler integration is completed). Using chordlength as a parametrization of the vortex sheet, the new values, $\Gamma(z_i(t+\Delta t))$, may be obtained by the interpolation of $\Gamma(z_{2i})$ (or $\Gamma(z_{1i})$), $\{\lambda_i\}$ and $\{\lambda_{si}\}$ and thus $\gamma(z_i(t+\Delta t))$ is calculated as described above.

The accuracy of the method was tested on the motion of the vortex sheet created by the irrotational flow circulation around a stagnant circular core. Moore & Griffith-Jones (1974) have shown that the flow is unstable. If a is the radius of the circular sheet and Γ the total circulation around the stagnant core, the undistributed potential

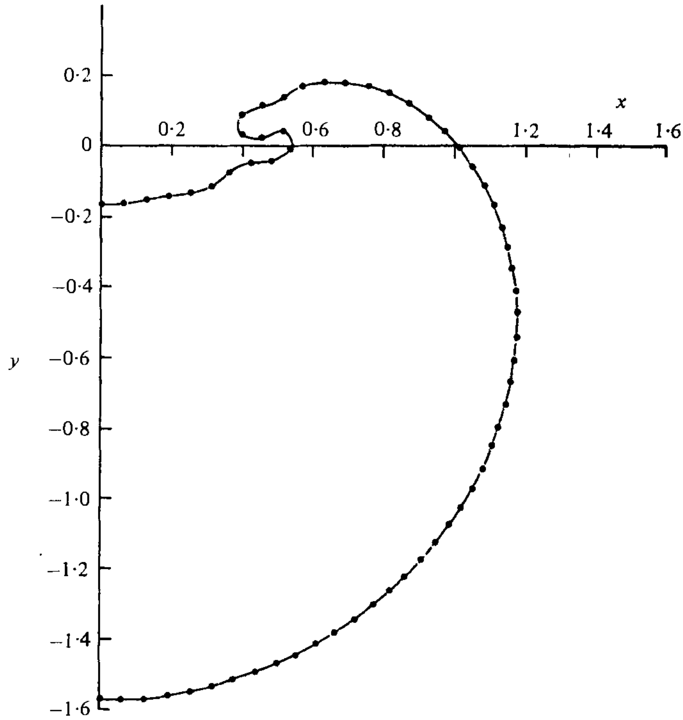

FIGURE 2. The vortex sheet behind a ring wing at $t = 1{\cdot}55$ with 61 mesh points along the sheet.

is $\phi = \Gamma\theta/2\pi$ for $r > a$, and $\phi = 0$ for $r \leqslant 0$. Consider perturbations in the position of sheet, $r = \eta(\theta, t)$, of the form

$$\eta = a + \epsilon \cos(n\theta + wt), \tag{3.6}$$

and perturbations in the potential of the form

$$\phi = \frac{\pi}{2\pi} + \frac{B}{r^n} \sin(n + wt) \quad \text{for} \quad r > \eta, \tag{3.7}$$

$$\phi = Ar^n \sin(n\theta + nt) \quad \text{for} \quad r < \eta. \tag{3.8}$$

After linearizing, the dispersion relation is obtained as

$$w^2 + w\frac{\pi}{2\pi a^2} + \frac{\Gamma}{8\pi^2 a^4} n(n-1) = 0. \tag{3.9}$$

Two modes were tested, $n = 1$, $w = 0$, and $n = 2$, $w = -\Gamma/2a^2$, which are both stable. For these cases, $A = 0$, $\epsilon\pi/4\pi a^3$ and $B = \epsilon\Gamma/2\pi$, $\epsilon\Gamma a/4$, respectively. It was sufficient to calculate only a few steps to check that the behaviour of the error with changes in the arclength spacing and the time-step show the correct behaviour. Provided the velocity was calculated sufficiently accurately the error was $O(\Delta t)$ for simple Euler integration whereas for modified Euler integration it was extremely small (round-off errors only were observed) since that scheme is exact for circular motion.

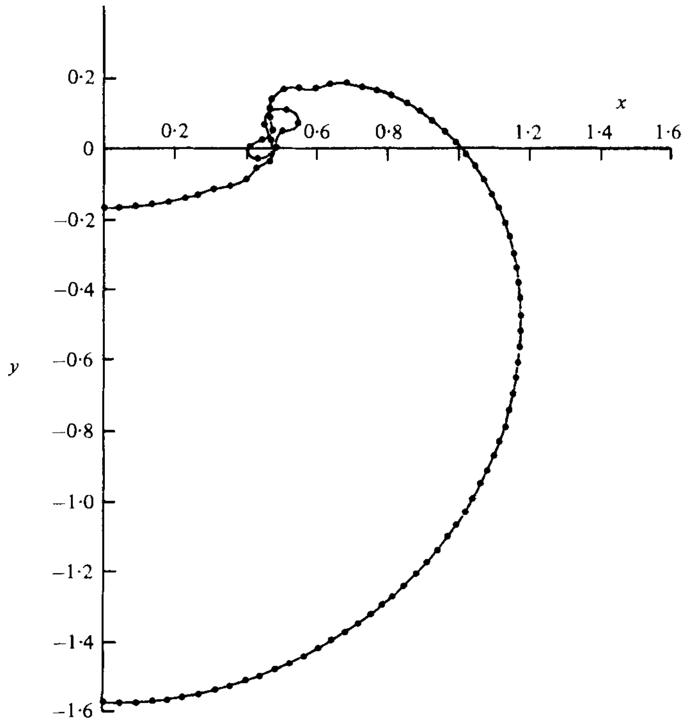

FIGURE 3. The vortex sheet behind a ring wing at $t = 1{\cdot}55$ with 91 mesh points along the sheet.

A far more interesting test of the method is the study of the motion of a vortex sheet shed by the ring wing. The vortex sheet is expected to roll up into two spirals and this behaviour has been observed experimentally by Bofah (1975). Suitable non-dimensional variables may be chosen so that initially the vortex sheet is circular with unit radius and the vortex sheet strength is $\gamma(\theta) = \cos\theta$. Since the flow has a plane of symmetry it is possible to follow only half the number of points required to represent the complete sheet. Interpolation also may be done on only half the sheet using end-point conditions that are dictated by the symmetry. The modified Euler formula and spline interpolation were used.

The numerical technique fails to produce good results. The calculation proceeds smoothly until roll-up begins; inevitably the sheet crosses itself. Moreover, as the number of points representing the sheet is increased, the breakdown in results occurs sooner. Figures 1–3 show the vortex sheet at $t = 1{\cdot}55$ with N, the number of points, of 41, 61 and 91. The time-step was 0·0194; smaller time-step did not change the results. The representation of the vortex sheet deteriorates when more points are used, contrary to the expectation expressed in the results of the error analysis. The corresponding profiles in figure 4 of the circulation measured along the sheet from $\theta = \frac{1}{2}\pi$ as functions of the arclength show an oscillatory behaviour. Bearing in mind that the numerical calculations show an increase in arclength with increasing N, the oscillations occur along the upper branch of the spiral. The nature of the oscillations is such that $\gamma(s)$ becomes negative, which is unphysical.

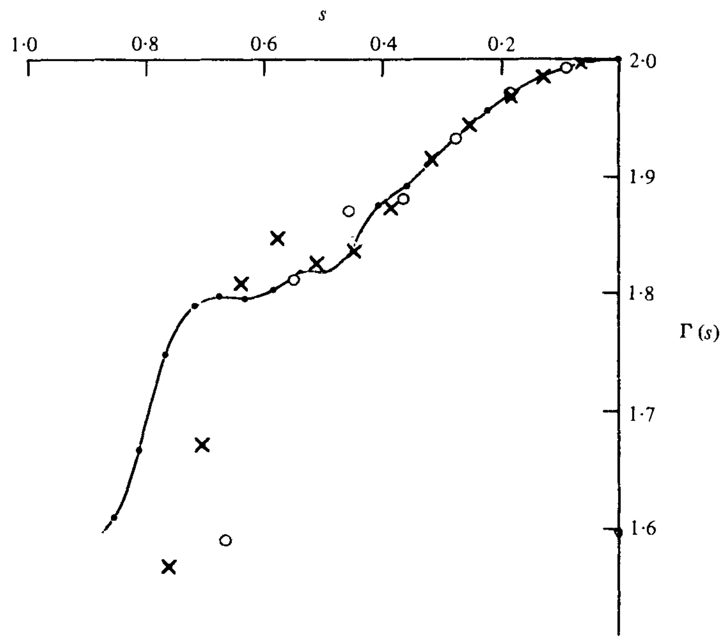

FIGURE 4. The circulation $\Gamma(s)$ along the sheet as a function of arclength s measured from the higher intercept point on the y axis for different numbers of mesh points: ○, 40 points; ×, 60 points; ——, 90 points.

In an attempt to determine what causes this numerical breakdown, different ways of implementing the method were tried, for example the less accurate scheme used by Fink & Soh (1978). Lagrange interpolation was used instead of spline interpolation and the points were evenly spaced in chordlength instead of arclength but the description of the sheet still deteriorated with increasing N.

4. Discussion of results

The nature of these results differ from those reported by Fink & Soh (1978) in their study of the roll-up of a sheet shed by an elliptically loaded wing. The reason is simple: Fink & Soh used a single vortex to represent the innermost turns of the spiral and the circulation of this vortex grows so large that it essentially dominates the flow field near the spiral. Their smooth results for a large number of points depend on this approximation (Baker 1977). On the other hand, the motion of a vortex sheet shed by a ring wing permits a direct study of their method.

Before exploring the possible reasons for the failure in obtaining reliable results, it is important to remember that questions concerning the existence or uniqueness of solutions to the equations describing vortex sheet motion have not been fully resolved. Recently Moore (1979) has presented mathematical arguments demonstrating the possibility that the problem is ill-posed. If this is the case, it is clear that viscosity must be incorporated and a thin layer of vorticity must replace the vortex sheet.

The stability of a vortex sheet is generally uncertain. The plane constant vortex sheet is known to be unstable (the Kelvin–Helmholtz instability), with disturbances having the smallest wavelengths growing the fastest. Small-scale instabilities may play an important role in the roll-up of vortex sheets (Pierce 1966). On the other hand, the effects of curvature and stretching of the sheet may be stabilizing (Moore 1976).

Thus it is possible that the difficulties experienced in following the vortex sheet numerically may be the consequence of the nature of the solutions to the equations describing vortex-sheet motion. If the vortex sheet shed by a ring wing is well defined and stable during its motion, the method is at fault. There are several possible ways whereby the method may prove inadequate.

One reason that the sheet crosses itself is that not enough points are used to resolve the structure of the sheet as different parts of it approach one another, leading to large errors in the calculation of the velocity (Maskew 1977). However, this cannot fully explain the deterioration in the description of the vortex sheet as the number of points is increased. Perhaps the vortex-sheet curvature changes rapidly in time, requiring a substantially larger number of points to fully describe the sheet. To test this idea proves too costly for the resources of this researcher. Instead a different numerical approach based on the 'cloud in cell' technique was used to explore the small-scale behaviour of the vortex sheet (Baker 1979). Small-scale instabilities generated by the numerical approximations rapidly destroyed the description of the sheet but the larger-scale motion is reproduceable and looks similar to the experimental results of Pierce (1966). Therefore, it is possible that small-scale motions do play a part in the roll-up of the sheets, and that Fink & Soh's method will be too costly to adequately account for them.

The numerical scheme may be unstable. Because of the complexity of the scheme it proves difficult to study its stability analytically. The basic method was tried on a simple one-dimensional model. The equation

$$\frac{\partial \Gamma}{\partial t} + \left(\frac{\partial \Gamma}{\partial s}\right)^2 = 0 \tag{4.1}$$

with the initial condition $\Gamma(s) = 1 - \cos(s)$ was solved by integrating

$$\frac{d\Gamma}{dt} = 0 \tag{4.2}$$

along the characteristics

$$\frac{ds}{dt} = \frac{d\Gamma}{ds}. \tag{4.3}$$

Thus equation (4.3) models equation (2.3). The results show smooth behaviour up to $t = \frac{1}{2}$, when the solution becomes multivalued. Of course, this is only a one-dimensional analogue and so curvature effects are ignored but it does give some reason to believe that the scheme is basically stable.

In fact the redistribution process can be expected to prevent small-scale instabilities from developing or at least to retard their growth. The redistribution process may induce other unphysical effects. This typically occurs when points are 'moved' through a larger distance by the redistribution process than by the velocity field. An example is given in Baker (1977) in which the roll-up of the sheet shed by an elliptically loaded wing, unwinds. Sarpkaya (1975) also report difficulties with the method, but no details are provided.

In conclusion, no definitive reason for the breakdown in result is known. Fink & Soh's (1978) technique is not reliable when used to study the roll-up of vortex sheets.

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